Stability of torsion free sheaves on curves and infinite-dimensional Grassmanian manifold

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In [**3**] and [**7**] are described the Krichever map from the set of torsion free sheaves on curves with some additional datas to an infinite-dimensional Grassmanian manifold. In this note we investigate the images of (semi)stable rank 2 torsion free sheaves on curves via the Krichever map and verify some analog of G.I.T. Hilbert-Mumford numerical criterion (theorem 2) with respect to actions of some one-parametric subgroups of the group $SL(2, k[[z]]$ on the determinant bundle of the infinite-dimensional Grassmanian manifold.

1 Krichever map

Consider the following geometric datas:

1. C is a reduced irreducible complete algebraic curve defined over a field k.
2. p is a smooth k-rational point.
3. $\mathcal{F}$ is a torsion free coherent sheaf of $\mathcal{O}_C$-modules on C of rank 2.
4. t_p is a local parameter of point p, i. e. $\hat{\mathcal{O}}_p = k[[t_p]]$, where $\hat{\mathcal{O}}_p$ is the completion of local ring $\mathcal{O}_p$ along the maximal ideal.
5. e_p ia a basis of rank 2 free module $\hat{\mathcal{F}}_p$ over the ring $\hat{\mathcal{O}}_p$, where $\hat{\mathcal{F}}_p \stackrel{\mathrm{def}}{=} \mathcal{F} \otimes_{\mathcal{O}_C} \hat{\mathcal{O}}_p$.

In the sequel we will *call* such collection $(C, p, \mathcal{F}, t_p, e_p)$ a *quintet*. If the Euler characteristic $\chi(\mathcal{F}) = \mu$, then we will say, that the quintet have an index μ.

On the other hand, consider $V = k((z)) \oplus k((z))$, $V_0 = k[[z]] \oplus k[[z]]$. Then *define* the infinite Grassmanian $Gr^\mu(V)$ as the set of k-vector subspaces W of V, which are Fredholm of index μ, i. e., $\dim_k V/(V_0 + W) < \infty$, $\dim_k W \cap V_0 < \infty$, and $\dim_k W \cap V_0 - \dim_k V/(V_0 + W) = \mu$.

There exists *the Krichever map* K from the set of quintets of index μ to $Gr^\mu(V)$, which can be shortly defined as the following chain of evident maps:

$$(C, p, \mathcal{F}, t_p, e_p) \to H^0(C \backslash p, \mathcal{F}) \hookrightarrow H^0(\mathrm{Spec}\, \mathcal{O}_p \backslash p, \mathcal{F}) \hookrightarrow H^0(\mathrm{Spec}\, \hat{\mathcal{O}}_p \backslash p, \mathcal{F}) =$$

$$= \hat{\mathcal{F}}_p \otimes_{\hat{\mathcal{O}}_p} \hat{K}_p \xrightarrow{e_p} (\hat{\mathcal{O}}_p \oplus \hat{\mathcal{O}}_p) \otimes_{\hat{\mathcal{O}}_p} \hat{K}_p \xrightarrow{t_p} k((z)) \oplus k((z)), \qquad (1)$$

where $\hat{K}_p$ is the fraction field of $\hat{\mathcal{O}}_p$.

The Krichever map K and its basic properties is considered in details by many authors. See, for example, [**3**] for an algebraic description. (But note that the definition in [**3**] is slightly different from our definition.) And see [**7**] for an analytic description.

For any $W \in Gr^{\mu}(V)$ define the ring

$$A_W \stackrel{\text{def}}{=} \{f \in k((z)) : fW \subset W\}$$

Lemma 1 (see [3]) *For any $W \in Gr^{\mu}(V)$ and any subring $A \subset A_W$ such that $k \subset A$ we have*

1) $A \cap k[[z]] = k$ and

2) if $A \neq k$, then the ring A has dimension 1 over k.

Proof . The first statement follows from $\dim_k W \cap V_0 < \infty$.

For the proof of the second statement consider the following subgroup of $\mathbb{Z}$:

$$\{n \in \mathbb{Z} \ : \ n = \nu(a)\,,\ a \in \operatorname{Frac} A\}\,,$$

where ν is the discrete valuation of $k((z))$ and $\operatorname{Frac} A$ is the fraction field of the ring A. Then this subgroup is $r\mathbb{Z}$ for some integer r. Therefore

1) there exist $f, g \in A$ such that $r = \nu(f) - \nu(g)$

2) for any integer n

$$\dim_k A \cap z^{-nr} k[[z]] / A \cap z^{(-n+1)r} k[[z]] \leq 1 \tag{2}$$

Now from (2), the first statement of this lemma, and $r = G.C.D.(\nu(f), \nu(g))$ we have

$$\dim_k A/k[f,g] < \infty \quad .$$

Thus the dimension of the ring A over k is at most two. Now let us assume that f and g are algebraically independent elements over k. Then for all pairs of integers $n_1 > 0$ and $n_2 > 0$ elements $f^{n_1} g^{n_2}$ are included in one basis of vector space A over k. But it contradicts to (2) and this contradiction concludes the proof of lemma.

Notice also the following well-known statement.

Lemma 2 *$W \in Gr^{\mu}(V)$ is in the image of the Krichever map if and only if* $\operatorname{rank} A_W = 1$*, i.e., there exist two elements $f, g \in A_W$ such that $\nu(f)$ and $\nu(g)$ are relatively prime.*

Proof . Note that the condition $\operatorname{rank} A_W = 1$ is equivalent to

$$\dim_k k((z))/(A_W + k[[z]]) < \infty \quad . \tag{3}$$

Let $W = K(C, p, \mathcal{F}, t_p, e_p)$. By A_C denote the image of the ring $H^0(C \backslash p, \mathcal{O}_C)$ in $k((z))$:

$$H^0(C \backslash p, \mathcal{O}_C) \hookrightarrow \hat{K}_p \xrightarrow{t_p} k((z)) \quad .$$

From the Riemann-Roch theorem we have

$$\dim_k k((z))/(A_C + k[[z]]) < \infty \quad .$$

From $A_C \subset A_W$ it follows (3).

Now let $\operatorname{rank} A_W = 1$. Fix any subring $A \subset A_W$ such that $k \subset A$ and $\dim_k A_W/A < \infty$. By lemma 1 the ring A has dimension 1 over k. Then from graded k-algebra

$$A_* \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} A \cap z^{-i}k[[z]]$$

we construct the complete irreducible reduced curve $C = \operatorname{Proj} A_*$. And from (3) we see that the graded A_*-module

$$W_* \stackrel{\text{def}}{=} \bigoplus_{i=-\infty}^{\infty} W \cap z^{-i}V_0$$

determines the rank 2 torsion free coherent sheaf $W_*\tilde{}$ on C. Moreover, we have $C = \operatorname{Spec} A \cup p$, where p is a smooth k-rational point, $\hat{\mathcal{O}}_p = k[[z]]$, $\mathcal{F}\mid_{C\backslash p} = W\tilde{}$, $\hat{\mathcal{F}}_p = V_0$, $t_p = z$, $e_p = (1,0),(0,1) \in V$. And

$$\bigoplus_{i=0}^{\infty} H^0(C, \mathcal{O}(ip)) \simeq A_* \qquad , \qquad \bigoplus_{i=-\infty}^{\infty} H^0(C, \mathcal{F}(ip)) \simeq W_* \quad \text{, and}$$

$$H^0(C, \mathcal{F}) \simeq W \cap V_0 \qquad , \qquad H^1(C, \mathcal{F}) \simeq V/(W + V_0) \quad .$$

This lemma is proved.

Remark. If we fix a triplet (C, p, t_p) and identify $(C, p, \mathcal{F}, t_p, e_p)$ with $(C, p, \mathcal{F}', t_p, e'_p)$ for every sheaf isomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ such that $\alpha(e_p) = e'_p$; then the Krichever map is an injective map. In this case $W \in Gr^{\mu}(V)$ is in the image of such map if and only if $A_C W \subset W$.

Now let $W = K(C, p, \mathcal{F}, t_p, e_p) \in Gr^{\mu}(V)$. Then with every subsheaf $\mathcal{G}$ of $\mathcal{F}$ we can associate the $k((z))$-vector subspace $L_{\mathcal{G}} \subset V$ such that $L_{\mathcal{G}} \cap W \neq 0$:

$$H^0(C\backslash p, \mathcal{G}) \hookrightarrow H^0(C\backslash p, \mathcal{F}) = W \subset Gr^{\mu}(V)$$

$$\mathcal{G} \longmapsto H^0(C\backslash p, \mathcal{G}) \cdot k((z)) \subset V \tag{4}$$

We have the following proposition.

Proposition 1 *Let $W = K(C, p, \mathcal{F}, t_p, e_p)$, $R(\mathcal{F})$ be the set of all rank 1 coherent subsheaves $\mathcal{G} \subset \mathcal{F}$ such that the sheaf $\mathcal{F}/\mathcal{G}$ is a free torsion coherent sheaf, $R(W)$ be the set of all 1-dimensional $k((z))$-vector subspaces $L \subset V$ such that $L \cap W \neq 0$. Then map (4) is an one-to-one correspondence between $R(\mathcal{F})$ and $R(W)$.*
Moreover, if $\mathcal{G} \in R(\mathcal{F}) \mapsto L_{\mathcal{G}} \in R(W)$, then

$$\chi(C, \mathcal{G}) = \dim_k W \cap L_{\mathcal{G}} \cap V_0 - \dim_k L_{\mathcal{G}}/(W \cap L_{\mathcal{G}} + L_{\mathcal{G}} \cap V_0) \quad . \tag{5}$$

Proof .

It is not difficult to see that the set $R(\mathcal{F})$ is in an one-to-one correspondence with 1-dimensional $k(\eta)$-vector subspaces of $\mathcal{F}_\eta \stackrel{\rm def}{=} H^0(\mathrm{Spec}\, k(\eta), \mathcal{F})$, where $\eta \hookrightarrow C$ is the general point. We have the canonical imbedding of $\mathcal{F}_\eta$ into V as the third term $H^0(\mathrm{Spec}\,\mathcal{O}_p \backslash p, \mathcal{F}) = \mathcal{F}_\eta$ of chain (1). And this imbedding gives us a one-to-one correspondence between $R(\mathcal{F})$ and $R(W)$.

Moreover, we have the following explicit construction. Let $L \in R(W)$. Then the graded $(A_C)_*$-module $(L \cap W)_* \stackrel{\rm def}{=} \bigoplus\limits_{i=-\infty}^{\infty} (L \cap W) \cap z^{-i} V_0$ determines the rank 1 torsion free sheaf $\mathcal{G}_L = (L \cap W)_*\tilde{}$ on $C = \mathrm{Proj}(A_C)_*$. Then $L \to \mathcal{G}_L$ inverts map (4) and

$$H^0(C, \mathcal{G}_L) \simeq W \cap L \cap V_0 \quad ,$$

$$H^1(C, \mathcal{G}_L) \simeq L/(W \cap L + L \cap V_0) \quad .$$

2 Determinant bundle

As was explicitly described in [**1**], $Gr^\mu(V)$ admits a structure of an algebraic scheme, which represents a functor, which generalizes the usual finite-dimensional grassmanian. There exists an open covering of $Gr^\mu(V)$ by means of $Gr^\mu_A(V)$, where A is a commensurable with V_0 k-vector subspaces in V, i. e., $\dim_k (A + V_0)/(A \cap V_0) < \infty$ and

$$Gr^\mu_A(V) \stackrel{\rm def}{=} \{W \in Gr^\mu(V) : V = A \oplus W\}$$

is an infinite-dimensional affine space which is isomorphic to the spectrum of a polynomial ring of infinitely many variables. One can define a linear bundle (determinant bundle) Det on $Gr^\mu(V)$ such that for any $W \in Gr^\mu(V)$

$$Det_W = \bigwedge^{max} (W \cap V_0) \otimes \bigwedge^{max} (V/W + V_0)^* \qquad .$$

Now let us describe the determinant bundle Det more explicitly ("in coordinats"). For this goal we identify the k-vector spaces V and $k((t))$ by means of the following continuous isomorphism:

$$\begin{gathered} V \longrightarrow k((t)) \\ (\sum a_i z^i) \oplus (\sum b_i z^i) \longmapsto \sum_i (a_i t^{2i} + b_i t^{2i+1}) \end{gathered} \tag{6}$$

Note that $V_0 \to k[[t]]$.

Let S_μ be the set of sequences $\{s_{-\mu+1}, s_{-\mu+2}, \ldots\}$ of integers such that:

1) these sequences are strictly decreasing;

2) $s_k = -k$ for $k \gg 0$.

Note that for every $\mu \in \mathbb{Z}$ the set S_μ is in an one-to-one correspondence with the Young diagramms.

For every $S = \{s_{-\mu+1}, s_{-\mu+2}, \ldots\} \in S_\mu$ let

$$H_S \in Gr^\mu(V)$$

be the k-vector space generated by $\{t^{s_k}\}$,

$$A_S \subset k((t))$$

be the smallest closed k-vector space generated by $\{t^q \mid q \in \mathbb{Z}\backslash S\}$. Then

$$Gr^\mu(V) = \bigcup_{S \in S_\mu} Gr^\mu_{A_S}(V) \quad . \tag{7}$$

For any $S \in S_\mu$ let $p_S : k((t)) \to H_S$ be the projection from the decomposition $H_S \oplus A_S = k((t))$. By $0(\mu) \in S_\mu$ denote $\{\mu-1,\ \mu-2,\ \mu-3,\ \ldots\}$.

We are now going to describe the determinant bundle Det in other words (see [**6**, §7.7]). Given $W \in Gr^\mu(V)$ let us define an *admissible* isomorphism to be an isomorphism

$$w : H_{0(\mu)} \longrightarrow W \qquad\qquad (w(t^{\mu-1}) \in W,\ w(t^{\mu-2}) \in W, \ldots)$$

such that

$$p_{0(\mu)} \cdot w - Id \quad : \quad H_{0(\mu)} \to H_{0(\mu)}$$

has finite rank. (Note that for any $W \in Gr^\mu(V)$ we can always find the corresponding admissible isomorphism.) Then the determinant bundle Det consists of

$$\{[\lambda, w] : \lambda \in k,\ w \text{ is an admissible isomorphism}\} \quad ,$$

which is identified with respect to the following equivalence relation:

$$[\lambda, w] \sim [\lambda \cdot \det(w'^{-1} \cdot w), w'] \quad ,$$

where w' and w are two admissible isomorphisms for $W \in Gr^\mu(V)$. (Note that

$$w'^{-1} \cdot w - Id \quad : \quad H_{0(\mu)} \to H_{0(\mu)}$$

has finite rank as an endomorphism of $H_{0(\mu)}$, and hence $\det(w'^{-1} \cdot w)$ is well defined with respect to the basis: $\{t^{\mu-1}, t^{\mu-2}, t^{\mu-3}, \ldots\}$.)

Let Det^* be the dual vector bundle to Det. For every $S \in S_\mu$ we can construct the global section

$$\pi_S \in H^0(Gr^\mu(V), Det^*) \qquad : \quad Det \longrightarrow k$$

$$[\lambda, w] \xrightarrow{\pi_S} \lambda \det(p_S \cdot w) \in k \quad ,$$

where $p_S \cdot w - Id \ : \ H_0 \to H_S$ has finite rank. Therefore we can compute $\det(p_S \cdot w)$ with respect to basises

$$\{t^{\mu-1},\ t^{\mu-2},\ \ldots\} \in H_{0(\mu)} \qquad \text{and} \qquad \{t^{s_{-\mu+1}},\ t^{s_{-\mu+2}},\ \ldots\} \in H_S \quad .$$

And as proven in [**1**] and [**6**], we have a closed imbedding ("Plücker imbedding"):

$$Gr^\mu(V) \hookrightarrow \mathbb{P}(\Pi(S_\mu)^*) \stackrel{\text{def}}{=} \mathrm{Proj}(Sym(\Pi(S_\mu)))$$

$$W \longmapsto \left\{\pi_S \to \pi_S(w) \stackrel{\text{def}}{=} \det(p_S \cdot w)\right\} \quad ,$$

where $\Pi(S_\mu) \subset H^0(Gr^\mu(V), Det^*)$ is the k-vector subspace generated by the global sections $\{\pi_S \ : \ S \in S_\mu\}$, and with every k-point $W \in Gr^\mu(V)$ we associate the 1-dimensional k-vector subspace $k \cdot \{\pi_S \mapsto \pi_S(w) \ : \ S \in S_\mu\} \subset \Pi(S_\mu)^*$, which is the same for all admissible isomorphisms w of W.

Note also that $\pi_S(w) \neq 0$ if and only if $W \in Gr^\mu_{A_S}(V)$, i. e. $W \oplus A_S = V$.

Consider the group

$$GL(V, V_0) \stackrel{\text{def}}{=} \{g \in Aut_k(V) \ : \ \dim_k(gV_0 + V_0)/(gV_0 \cap V_0) < \infty\} \quad .$$

Note that "in coordinats" $GL(V,V_0)$ corresponds to the group of invertible elements of the algebra of matrices $(A_{ij})_{i,j\in\mathbb{Z}}$ such that for every integer l the number of non-zero A_{ij} with $j \geq l$, $i \leq l$ is finite. And the action of $(A_{ij})_{i,j\in\mathbb{Z}}$ on $k((t))$ is $E_{ij}(t^j) = t^i$. (Here E_{ij} is the matrix with a 1 on the (i,j)-th entry and zeros elsewhere.) Define the subgroup

$$GL_0(V, V_0) \stackrel{\text{def}}{=} \{g \in GL(V, V_0) \ : \ \dim_k gV_0/(gV_0 \cap V_0) = \dim_k V_0/(gV_0 \cap V_0)\} \quad .$$

Then we have the obvious action of $GL_0(V,V_0)$ on $Gr^\mu(V)$. Let the group $\hat{GL}_0(V,V_0)$ be the central extension of the group $GL_0(V,V_0)$ by means of k^*, which is the group of all automorphisms of linear bundle Det on $Gr^0(V)$ which cover actions of elements of group $GL_0(V,V_0)$ on $Gr^0(V)$. There exists an explicit description of $\hat{GL}_0(V,V_0)$ and its action on $Det\mid_{Gr^\mu(V)}$ (see [**6**]). Let H be the group

$$H = \Big\{(g,E) \in GL_0(V,V_0) \times GL_k(t^{-1}k[t^{-1}]) \quad : \quad g_{--} - E \ : \ t^{-1}k[t^{-1}] \to t^{-1}k[t^{-1}] \quad \text{has finite rank}\Big\},$$

where $g_{--} \stackrel{\text{def}}{=} (g)_{i,j\leq -1}$. Let $N \subset H$ be the normal subgroup defined as $N = \{(1,E) \in H \ : \ \det E = 1\}$. Then $\hat{GL}_0(V,V_0) = H/N$, and the projection on the first factor gives us the map onto $GL_0(V,V_0)$. Now describe the action of $\hat{GL}_0(V,V_0)$ on $Det\mid_{Gr^\mu(V)}$.

$\underline{\mu = 0}$: for $(g,E) \in H$ and $[\lambda, w] \in Det\mid_{Gr^0(V)}$

$$(g,E)[\lambda,w] \stackrel{\text{def}}{=} [\lambda, gwE^{-1}] \tag{8}$$

This expression gives us the correct action of $\hat{GL}_0(V,V_0)$ on $Det\mid_{Gr^0(V)}$.

$\underline{\mu \neq 0}$. Define the action $a \in \hat{GL}_0(V,V_0)$ on $Det\mid_{Gr^\mu(V)}$ as action of

$$\sigma^{-\mu} \cdot \tilde{\sigma}^\mu(a) \cdot \sigma^\mu \quad , \tag{9}$$

where $\sigma : Det\mid_{Gr^\mu(V)} \to Det\mid_{Gr^{\mu-1}(V)}$ is defined by the formula $\sigma \cdot [\lambda, w] = [\lambda, t^{-1} \circ w]$; $t^{-1} \circ w \ : \ H_{0(\mu-1)} \to t^{-1}W$, $t^{-1} \circ w(x) = t^{-1}w(tx)$ is admissible, $x \in H_{0(\mu-1)}$; and the

automorphism $\tilde{\sigma} \; : \; G\hat{L}_0(V,V_0) \to G\hat{L}_0(V,V_0)$ is induced by the following endomorphism of H:

$$(g,E) \longmapsto (t^{-1}gt, E_\sigma) \quad \text{, where}$$

$$E_\sigma \mid_{t^{-2}k[t^{-1}]} = t^{-1}Et \qquad \text{and} \qquad E_\sigma(t^{-1}) = 1 \quad .$$

Now consider the group $\Gamma = GL(2, k[[z]])$, which is the subgroup of $GL_0(V,V_0)$.

Lemma 3 *The central extension $G\hat{L}_0(V,V_0)$ splits over Γ.*

Proof . Define a group homomorphism

$$h \; : \; \Gamma \longrightarrow H \quad \text{by} \quad h(\gamma) = (\gamma, \gamma_{--}) \quad \text{for} \quad \gamma \in \Gamma \, . \tag{10}$$

(Observe that Γ keeps V_0 stable and hence $\gamma_{--} \in GL_k(k[t^{-1}])$). Then the group homomorphism

$$\Gamma \xrightarrow{h} H \to H/N = G\hat{L}_0(V,V_0)$$

splits the central extension over Γ.

In $V = k((z)) \oplus k((z))$ consider two lines:

$$l_1 = k((z)) \oplus 0 \qquad \text{and} \qquad l_2 = 0 \oplus k((z)) \quad ,$$

which correspond to $k((t^2))$ and $tk((t^2))$ in $k((t))$. For any $S \in S_\mu$ let

$$n_i(S) \stackrel{\text{def}}{=} \dim_k H_S \cap V_0 \cap l_i - \dim_k l_i/(H_S \cap l_i + l_i \cap V_0) \; , \quad i = 1,2$$

Note that

$$n_1(S) + n_2(S) = \mu \quad . \tag{11}$$

Let $\begin{pmatrix} \alpha^{a_1} & 0 \\ 0 & \nu^{a_2} \end{pmatrix} \in \Gamma$, $\alpha, \; \nu \in k^* \quad a_i \in \mathbb{Z}$. By lemma 3 we can consider an action of $\begin{pmatrix} \alpha^{a_1} & 0 \\ 0 & \nu^{a_2} \end{pmatrix}$ on the determinant bundle Det.

Proposition 2 *Let $S \in S_\mu$, let π_S be the corresponding global section of $Det^* \mid_{Gr^\mu(V)}$. Let $\begin{pmatrix} \alpha^{a_1} & 0 \\ 0 & \nu^{a_2} \end{pmatrix}^*$ be the action of $\begin{pmatrix} \alpha^{a_1} & 0 \\ 0 & \nu^{a_2} \end{pmatrix}$ on $H^0(Gr^\mu(V), Det^*)$. Then*

$$\begin{pmatrix} \alpha^{a_1} & 0 \\ 0 & \nu^{a_2} \end{pmatrix}^* (\pi_S) = \alpha^{a_1 \cdot n_1(s)} \nu^{a_2 \cdot n_2(s)} \pi_S \quad .$$

Proof follows from an explicit description of actions of $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}$ on the determinant bundle. We get this description by means of direct calculations from formulas (8), (9) and (10).

For any $S \in S_\mu$ let $n(S) \stackrel{\text{def}}{=} n_1(S) - n_2(S)$.

Lemma 4 *Let* $W \in Gr^\mu(V)$. *Let* $S(W) = \{S \in S_\mu \ : \ \pi_S(W) \neq 0\}$. *Then we have*

1. *if* $\min\limits_{S \in S(W)} n(S) > -\infty$, *then* $l_1 \cap W \neq 0$;
2. *if* $\operatorname{rank} A_W = 1$, $l_1 \cap W \neq 0$, *then* $\min\limits_{S \in S(W)} n(S) > -\infty$.

Proof .

Remind that

$$\pi_S(W) \neq 0 \quad \text{if and only if} \quad W \oplus A_S = V \tag{12}$$

Consider the second statement of this lemma. If $l_1 \cap W \neq 0$, then from $\dim_k W \cap V_0 < \infty$ we have $\dim_k W \cap l_1 \cap V_0 < \infty$. And from $\operatorname{rank} A_W = 1$ we have

$$\dim_k k((z))/(A_W + k[[z]]) < \infty \quad \text{, therefore}$$

$$\dim_k l_1/(W \cap l_1 + V_0 \cap l_1) < \infty \quad .$$

Now let some $S' \in S(W)$, then $W \oplus A_{S'} = V$. Hence $W \cap A_{S'} = 0$. Hence

$$(W \cap l_1) \cap (A_{S'} \cap l) = 0 \quad . \tag{13}$$

From (13) we obtain

$$n_1(S') \geq \dim_k W \cap l_1 \cap V_0 - \dim_k l_1/(W \cap l_1 + V_0 + l_1) \quad .$$

From (11) we have $n(S') = 2n_1(S') - \mu$. Therefore

$$n(S') \geq 2\left(\dim_k W \cap l_1 \cap V_0 - \dim_k l_1/(W \cap l_1 + V_0 \cap l_1)\right) - \mu \quad . \tag{14}$$

The second statement of this lemma is proved.

Now consider the first statement of this lemma. From $\min\limits_{S \in S(W)} n(S) > -\infty$ we see that $\min\limits_{S \in S(W)} n_1(S) > -\infty$. Let

$$\tau = \min(0, \min_{S \in S(W)} n_1(S)) - 1 \quad .$$

For any integer $m > 0$ define

$$B_m = z^\tau k[[z]] \oplus z^{m+\mu} k[[z]] \subset V \quad .$$

Fix some integer $m_0 > \max(0, -\mu, -\tau)$ such that

$$W \cap (z^{m_0+\mu} V_0) = 0 \qquad \text{and} \qquad V/(W + z^{-m_0} V_0) = 0 \ .$$

We will show that for every $m > m_0$

$$W \cap B_m \neq 0 \ . \tag{15}$$

For this goal we consider the images $\tilde{W}$ of $W \cap (z^{-m} V_0)$ and $\tilde{l_1}$ of $l_1 \cap (z^{-m} V_0)$ in the finite-dimensional Grassmanian manifold

$$Gr(2m + \mu, z^{-m} V_0 / z^{m+\mu} V_0) \ .$$

Now (15) is equivalent to

$$(\tilde{W} \cap \tilde{l_1}) \cap ((z^{\tau} V_0 / z^{m+\mu} V_0) \cap \tilde{l_1}) \neq 0 \ .$$

And the last formula follows from comparison of dimensions of finite-dimensional vector spaces:

$$\dim_k \tilde{l_1} = 2m + \mu$$

$$\dim_k (z^{\tau} V_0 / z^{m+\mu} V_0) \cap \tilde{l_1} = m + \mu - \tau$$

$$\dim_k \tilde{W} \cap \tilde{l_1} > m + \tau \quad , \tag{16}$$

where (16) is true by the following reason.
If

$$\dim_k \tilde{W} \cap \tilde{l_1} \leq m + \tau \quad , \tag{17}$$

then we find a k-vector subspace $K \subset \tilde{l_1}$ such that K is the span of $\{z^{i_k}\}$ for some integers i_k and $\tilde{l_1} = (\tilde{W} \cap \tilde{l_1}) \oplus K$. Then we can find $\tilde{L} \in Gr(2m + \mu, z^{-m} V_0 / z^{m+\mu} V_0)$ such that $\tilde{L}$ is the span of $\{z^{i_l}\}$ for some integers i_l, $\tilde{L} \cap \tilde{l_1} = K$, and

$$z^{-m} V_0 / z^{m+\mu} V_0 = \tilde{W} \oplus \tilde{L} \quad .$$

Let L be the pre-image of $\tilde{L}$ in $z^{-m} V_0$. We can find the Plücker coordinate $S' \in S_{\mu}$ such that $L = A_{S'}$. Then, by construction,

$$V = W \oplus A_{S'} \quad , \tag{18}$$

and from (17) $n_1(S') \leq \tau < \min\limits_{S \in S(W)} n_1(S)$. The last formula contradicts (18). Therefore formula (16) is proved. Hence formula (15) is true.

Now from the Fredholm condition on W we have $\dim_k W \cap (z^{\tau} V_0) < \infty$. From (15) we have non-empty filtration $W \cap B_m$ for all sufficiently large integers m. The last one is equivalent to

$$(W \cap (z^{\tau} V_0)) \cap l_1 \neq 0 \quad .$$

Indeed, assume the converse. Then choose a finite k-basis $e_1, \ldots, e_n$ of $W \cap z^{\tau} V_0$ such that $\nu_2(e_i) \neq \nu_2(e_j)$ for all pairs i, j. (Here $\nu_2(e_k) = p$ if $e_k \in B_p$ but $e_k \neq B_{p+1}$.) Then for any $m > \max\limits_i \nu_2(e_i)$ we have $W \cap B_m = 0$. This contradiction concludes the proof of lemma 4.

Lemma 5 *Let* $W \in Gr^{\mu}(V)$ *,* $\operatorname{rank} A_W = 1$ *, and condition 1 or condition 2 of the previous lemma is true, then*

$$\min_{S \in S(W)} n(S) = 2\left(\dim_k W \cap l_1 \cap V_0 - \dim_k l_1/(W \cap l_1 + V_0 \cap l_1)\right) - \mu \qquad (19)$$

Proof . From the proof of expression (14) we have that the left hand side of expression (19) is greater or equal to the right hand side of (19).

Now fix some integer $m > 0$ such that

$$W \cap (z^{m+\mu} V_0) = 0 \qquad \text{and} \qquad V/(W + z^{-m} V_0) = 0 \ .$$

Consider the images $\tilde{W}$ of $W \cap (z^{-m} V_0)$ and $\tilde{l_1}$ of $l_1 \cap (z^{-m} V_0)$ in $z^{-m} V_0 / z^{m+\mu} V_0$.

Then find a k-vector subspace $K \subset \tilde{l_1}$ such that K is the span of $\{z^{i_k}\}$ for some integers i_k and

$$\tilde{l_1} = (\tilde{W} \cap \tilde{l_1}) \oplus K \quad . \qquad (20)$$

Then we can find $\tilde{L} \subset z^{-m} V_0 / z^{m+\mu} V_0$ such that $\tilde{L}$ is the span of $\{z^{i_l}\}$ for some integers i_l , $\tilde{L} \cap \tilde{l_1} = K$, and

$$z^{-m} V_0 / z^{m+\mu} V_0 = \tilde{W} \oplus \tilde{L} \quad . \qquad (21)$$

Denote by L the pre-image of $\tilde{L}$ in $z^{-m} V_0$. Then there exists some $S' \in S_{\mu}$ such that $L = A_{S'}$ and $V = W \oplus A_{S'}$. From (20) and (21) we have $n(S') = n(W)$. Lemma 5 is proved.

3 Stable and non-stable points

For torsion free sheaves on C one can define the notion of "stability" and "semistability". (See [**4**].) Remind these definitions.

Definition 1 *A rank* 2 *torsion free sheaf* $\mathcal{F}$ *on a curve* C *is called a semistable sheaf if for any subsheaf* $\mathcal{G} \in R(\mathcal{F})$

$$2\chi(\mathcal{G}) \le \chi(\mathcal{F}) \quad .$$

Definition 2 *A rank* 2 *torsion free sheaf* $\mathcal{F}$ *on a curve* C *is called a stable sheaf if for any subsheaf* $\mathcal{G} \in R(\mathcal{F})$

$$2\chi(\mathcal{G}) < \chi(\mathcal{F}) \quad .$$

Theorem 1 *Let* $W = K(C, p, \mathcal{F}, t_p, e_p)$ *. Then the sheaf* $\mathcal{F}$ *is not a (semi)stable sheaf if and only if there exists some* $g \in SL(2, k[[z]])$ *such that for any* $S \in S(gW)$ $n(S) \ge 0$ (> 0) *.*

Proof follows from proposition 1, lemmas 2, 4 and 5, and facts that

1) group $SL(2,k[[z]])$ acts transitively on the set of all 1-dimensional over $k((z))$ vector subspaces in V;

2) the group $SL(2,k[[z]])$ keeps V_0 stable.

Remark. From lemma 5 and formula (5) we can also obtain numerical datas of nonstability of the sheaf.

Example 1. (See also [**3**, prop. 3.8].) Let $\mu = 2\nu$ for some integer ν, and $W = K(C,p,\mathcal{F},t_p,e_p)$ is a point of the big cell of Gr_V^{μ}, i. e.,

$$\pi_{0(\mu)}(W) \neq 0 \quad , \tag{22}$$

then the sheaf $\mathcal{F}$ is semistable. Indeed, if condition (22) is true, then $W \oplus z^{\nu}V_0 = V$ and for any $g \in SL(2,k[[z]])$, $g(z^{\nu}V_0) = z^{\nu}V_0$, $gW \oplus z^{\nu}V_0 = V$. Therefore $\pi_{0(\mu)}(gW) \neq 0$. But $n(0(\mu)) = 0$, and we apply theorem 1.

We can generalize example 1 in the following way.

Example 2. With every $W \in Gr^{\mu}(V)$ one can associate (see [**5**]):

1) the function $T_W : \mathbb{Z} \to \{0,1,2\}$ such that $T_W(i) = 2$ for $i \ll 0$, $T_W(i) = 0$ for $i \gg 0$, and $T_{gW} = T_W$ for any $g \in SL(2,k[[z]])$

$$T_W(i) \stackrel{\text{def}}{=} \dim_k (W \cap z^i V_0)/(W \cap z^{i+1}V_0) \ ,$$

2) the point

$$x_W = (x_{W,i_1}, \ldots, x_{W,i_{l_W}}) \in (\mathbb{P}^1_k)^{l_W} \quad , \tag{23}$$

where $i_1, \ldots, i_{l_W}$ are all the integers such that $T_W(i_j) = 1$, and the point $x_{W,i_j} \in \mathbb{P}^1_k$ is the 1-dimensional k-vector space $(W \cap z^{i_j}V_0)/(W \cap z^{i_j+1}V_0)$ in the 2-dimensional k-vector space $z^{i_j}V_0/z^{i_j+1}V_0$.

Note that all $\mathbb{P}^1_k$ from (23) have the canonical homogeneous coordinates, which are induced by $z^{i_j} \oplus 0 \in V$ and $0 \oplus z^{i_j} \in V$ for the corresponding j. By means of these coordinates we can identify all $\mathbb{P}^1_k$ from (23) and consider x_W as l_W ordered points on a line.

There exists the natural diagonal action of the group $SL(2,k)$ on $(\mathbb{P}^1_k)^{l_W}$. Consider the Segre imbedding $(\mathbb{P}^1_k)^{l_W} \hookrightarrow \mathbb{P}^{2l_W-1}_k$. There is a unique linear action of $SL(2,k)$ on $\mathbb{P}^{2l_W-1}_k$ which makes the Segre imbedding into a $SL(2,k)$-morphism (the l_W-fold tensor represantation of $SL(2,k)$). Also, $SL(2,k)$ is a reductive group. Therefore G.I.T. determines (semi)stable points on $(\mathbb{P}^1_k)^{l_W}$ with respect to the $SL(2,k)$ action and the Segre imbedding. From [**4**] we have the following explicit description: $x \in (\mathbb{P}^1_k)^{l_W}$ is a (semi)stable point if and only if no point of $\mathbb{P}^1_k$ occurs as a component of $x \geq l_W/2$ $(> l_W/2)$ times.

Now we can formulate the following proposition.

Proposition 3 *Let* $W = K(C, p, \mathcal{F}, t_p, e_p)$ *. Then*

1) If $1 \notin \operatorname{Im} T_W$ *, then* $\mathcal{F}$ *is a semistable sheaf.*

2) If $x_W \in (\mathbb{P}^1_k)^{l_W}$ *is a (semi)stable point with respect to the* $SL(2,k)$ *action and the Segre imbedding, then* $\mathcal{F}$ *is a (semi)stable sheaf.*

Proof .

Indeed, if $1 \notin \mathrm{Im} T_W$, then let $M \subset V$ be the span of $\Big\{\{z^i \oplus 0\}, \{0 \oplus z^i\} : T_W(i) = 2\Big\}$. We can find $S' \in S_\mu$ such that $M = H_{S'}$. By construction, $W \oplus A_{S'} = V$. Hence $\pi_{S'}(W) \neq 0$. For any $g \in SL(2, k[[z]])$ we have $T_{gW} = T_W$. Therefore $\pi_{S'}(gW) \neq 0$. From theorem 1 and $n(S') = 0$ we see that $\mathcal{F}$ is a semistable sheaf.

Now consider the case 2. Let $x_W = (x_{W,i_1}, \ldots, x_{W,i_{l_W}}) \in (\mathbb{P}^1_k)^{l_W}$. For every $k \in \{i_1, \ldots, i_{l_W}\}$ we can choose $z_k \in V$ equal to $\{z^k \oplus 0\}$ or $\{0 \oplus z^k\}$ such that the image of the k-line $k \cdot z_k$ in $z^k V_0 / z^{k+1} V_0$ does not coincide with $x_{W,k}$. Let $M \subset V$ be the span of $\Big\{z_k : k \in \{i_1, \ldots, i_{l_W}\}\Big\}$ and $\Big\{\{z^i \oplus 0\}, \{0 \oplus z^i\} \ : \ T_W(i) = 2\Big\}$. Then there exists $S' \in S_\mu$ such that $M = H_{S'}$ and $W \oplus A_{S'} = V$, i. e., $\pi_{S'}(W) \neq 0$. Moreover, if $x_W \in (\mathbb{P}^1_k)^{l_W}$ is a (semi)stable point, then from the explicit description of such points we can choose $z_k \in V$ such that

$$n(S') < 0 \ (\leq 0) \quad . \tag{24}$$

The action of $SL(2,k)$ on $(\mathbb{P}^1_k)^{l_W}$ is induced by the action of $SL(2, k[[z]])$ on V. Therefore $x_{gW} \in (\mathbb{P}^1_k)^{l_{gW}}$ is a (semi)stable point for any $g \in SL(2, k[[z]])$. Now from this fact, formula (24), and theorem 1 we get that $\mathcal{F}$ is a (semi)stable sheaf.

Remark. Suppose that $x \in (\mathbb{P}^1_k)^l$ is not a semistable point with respect to the $SL(2,k)$ action and the Segre imbedding. Then it is easy to construct quintets $(C, p, \mathcal{F}, t_p, e_p)$ with stable, semistable, and nonstable sheaves $\mathcal{F}$ such that for $W = K(C, p, \mathcal{F}, t_p, e_p)$ we have $l_W = l$ and $x_W = x$.

Remark. One can introduce the cellular decomposition of $Gr^\mu(V)$ indexed by the set S_μ (see [**6**]): $W \in Gr^\mu(V)$ belongs to the cell $S = \{s_{-\mu+1}, s_{-\mu+2}, \ldots\}$ if $\pi_S(W) \neq 0$ while $\pi_{S'}(W) = 0$ unless $s'_{-\mu+1} \geq s_{-\mu+1}, s'_{-\mu+2} \geq s_{-\mu+2}, \ldots$. (Such $S \in S_\mu$ is uniquely defined for each $W \in Gr^\mu(V)$.) It is not difficult to see that the case 1 of proposition 3 corresponds to the following statement: if $W = K(C, p, \mathcal{F}, t_p, e_p)$ belongs to the cell $S \in S_\mu$ such that $H_S \cap l_1 = H_S \cap l_2$, then $\mathcal{F}$ is a semistable sheaf.

Now note that the action of the group $SL(2, k[[z]])$ on $Gr^\mu(V)$ keeps $W = K(C, p, \mathcal{F}, t_p, e_p) \in Gr^\mu(V)$ in the image of the Krichever map and does not change the curve C, the point p, the local parameter t_p, and the sheaf $\mathcal{F}$. So this action is an action only on "basises e_p". Therefore if we want to get something like "moduli spaces" of sheafs on C, we would need "to kill" the action of $SL(2, k[[z]])$ on $Gr^\mu(V)$.

On the other hand, by lemma 3, the group $SL(2, k[[z]])$ acts on $Det \mid_{Gr^\mu(V)}$. Moreover, from this action we obtain an infinite-dimensional representation of $SL(2, k[[z]])$ in the k-vector space $\Pi(S_\mu)$. (In general, this representation is the restriction of representation of $\hat{GL}_0(V, V_0)$ in $\Pi(S_\mu)$, which can be got in the same way and is usually called the

infinite wedge-representation (see [**2**]).) Hence we have a linear action of $SL(2,k[[z]])$ on $\mathbb{P}(\Pi(S_\mu)^*)$. And by construction, this action is compatible with the action of $SL(2,k[[z]])$ on $Gr^\mu(V)$ via the Plücker imbedding. These arguments lead us to the following definitions.

For any $W \in Gr^\mu(V)$ denote by $\hat{W}$ some nonequal to 0 element from $\Pi(S_\mu)^*$ such that the image of $\hat{W}$ in $\mathbb{P}(\Pi(S_\mu)^*)$ coincides with the image of W via the Plücker imbedding.

Let

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2,k[[z]]),\ \lambda \in k^* \right\}$$

be a 1-parametric subgroup: $k^* \to SL(2,k[[z]])$. For any $g \in SL(2,k[[z]]$ define the group ${}^gT = gTg^{-1}$. Let ${}^gT(\lambda) = g \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} g^{-1}$ be an element from ${}^gT$.

It will be convinient to identify k^* with a subset of $\mathbb{P}^1_k$ in the obvious manner. In fact, for any $\lambda \in k$, we identify λ with the point $(1,\lambda)$ in $\mathbb{P}^1_k$, and write ∞ for the extra point $(0,1)$. Note that the morphism $k^* \to \Pi(S_\mu)^* \ : \ \lambda \mapsto {}^gT(\lambda)\hat{W}$ may or may not extend to the points 0, ∞, but if it does extend to either or both of these points the extension is unique. Thus we can define the expressions $\lim\limits_{\lambda\to 0} {}^gT(\lambda)\hat{W}$ and $\lim\limits_{\lambda\to\infty} {}^gT(\lambda)\hat{W}$ in an obvious way.

Definition 3 *A closed point $W \in Gr^\mu(V)$ is a semistable point with respect to the 1-parametric subgroup ${}^gT$ if for $x = 0$ and $x = \infty$ $\lim\limits_{\lambda\to x} {}^gT(\lambda)\hat{W}$ does not exist or $\lim\limits_{\lambda\to x} {}^gT(\lambda)\hat{W} \neq 0$.*

Definition 4 *A closed point $W \in Gr^\mu(V)$ is a stable point with respect to the 1-parametric subgroup ${}^gT$ if for $x = 0$ and $x = \infty$ $\lim\limits_{\lambda\to x} {}^gT(\lambda)\hat{W}$ does not exist.*

Now using these definitions and proposition 2 we can reformulate theorem 1 in the following way.(Compare [**4**].)

Theorem 2 *Let $(C,p,\mathcal{F},t_p,e_p)$ be a quintet, $\chi(\mathcal{F}) = \mu$. Then the sheaf $\mathcal{F}$ is a (semi)stable sheaf on the curve C if and only if the point $K(C,p,\mathcal{F},t_p,e_p) \in Gr^\mu(V)$ is a (semi)stable point with respect to the 1-parametric subgroups ${}^gT$ for all $g \in SL(2,k[[z]])$.*

Acknowledgments

This note was done during my visit to the Abdus Salam ICTP in the end of 1998. I am very much grateful to Professor M. S. Narasimhan for the invitation me to the ICTP and for the excellent working conditions. I am deeply grateful also to my scientific advisor Professor A. N. Parshin for the permanent attention and help.

References

[1] Álvarez Vázauez A., Muños Porras J. M., Plaza Martín F. J., *The algebraic formalism of soliton equations over arbitrary base fields*, preprint alg-geom/9606009.

[2] Kac V. G., Peterson D. H., *Lectures on the infinite wedge-representation and the MKP hierarchy*, seminare de Math. Superieures, Les Presses de L'Université de Montréal, 102 (1986), p.141-186.

[3] Mulase M.,*Category of vector bundles on algebraic curves and infinite dimensional grassmanians*, Intern. J. of Math. 1990. v.1 n.3 p.293-342.

[4] Newstead P. E., *Introduction to moduli problems and orbit spaces*, Tata Inst. of Fund. Research Lect. Notes 51. 1978.

[5] Parshin A. N., *Infinite Grassmanians and vector bundles*, to appear.

[6] Pressley A., Segal G., *Loop Groops*, Oxford University Press (1985).

[7] Previato E., Wilson G., *Vector bundles over curves and solutions of the KP equations*, Proc. Sympos. Pure Math., 49(1989), p.553-569.

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